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# SMALL PROGRAMMING EXERCISES 2

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The four exercises we present this time are all variations on a common theme. See if you can discover their underlying solution strategy. All exercises allow linear solutions, i.e. solutions whose computation times are proportional to the sizes of the arrays given. None of them require the introduction of auxiliary arrays. Exercise 5 is due to Jan Tijmen Udding. The other three exercises have been around for some time already. (For an exposition on the notation used the reader is referred to the previous issue of Small Programming Exercises.)

*Exercise 3: coincidence count*

Find a statement list $S$ such that

$$
\begin{array}{l}
|[M, N\colon \mathit{int}\{M \geqslant 0 \wedge N \geqslant 0\} \\
;\ F(i\colon 0 \leqslant i < M),\ G(j\colon 0 \leqslant j < N)\colon \textbf{array of } \mathit{int} \\
\qquad \{F \text{ and } G \text{ are increasing}\} \\
;\ |[c\colon \mathit{int} \\
\quad ;\ S \\
\quad \{c = (\mathbf{N} i,j\colon 0 \leqslant i < M \wedge 0 \leqslant j < N\colon F(i) = G(j))\} \\
\quad ]| \\
]|
\end{array}
$$

*Exercise 4: dominance count*

Find a statement list $S$ such that

$$
\begin{array}{l}
|[M, N\colon \mathit{int}\{M \geqslant 0 \wedge N \geqslant 0\} \\
;\ F(i\colon 0 \leqslant i < M),\ G(j\colon 0 \leqslant j < N)\colon \textbf{array of } \mathit{int} \\
\qquad \{F \text{ and } G \text{ are ascending}\} \\
;\ |[c\colon \mathit{int} \\
\quad ;\ S \\
\quad \{c = (\mathbf{N} i,j\colon 0 \leqslant i < M \wedge 0 \leqslant j < N\colon F(i) > G(j))\} \\
\quad ]| \\
]|
\end{array}
$$

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*Exercise 5: head and tail sums*

Find a statement list $S$ such that

$$
\begin{array}{l}
|[N\colon int\{N \geq 0\} \\
;F(i\colon 0 \leq i < N)\colon \textbf{array of } int\{\mathbf{A}i\colon 0 \leq i < N\colon F(i) > 0\}\} \\
;|[c\colon int \\
\quad ;S \\
\quad \{c = (\mathbf{N}j,k\colon 0 \leq j \leq N \wedge 0 \leq k \leq N\colon \\
\qquad\qquad (\mathbf{S}i\colon 0 \leq i < j\colon F(i)) = (\mathbf{S}i\colon k \leq i < N\colon F(i)))\} \\
\quad ]| \\
]|
\end{array}
$$

*Exercise 6: minimal distance*

Find a statement list $S$ such that

$$
\begin{array}{l}
|[M,N\colon int\{M \geq 1 \wedge N \geq 1\} \\
;F(i\colon 0 \leq i < M), G(j\colon 0 \leq j < N)\colon \textbf{array of } int \\
\quad \{F \text{ and } G \text{ are ascending}\} \\
;|[a\colon int \\
\quad ;S \\
\quad \{a = (\mathbf{MIN}\ i,j\colon 0 \leq i < M \wedge 0 \leq j < N\colon abs(F(i) - G(j)))\} \\
\quad ]| \\
]|
\end{array}
$$

*Solution of Exercise 1 (longest smooth segment)*

With $SS(p,q)$ denoting

$$(\mathbf{A}i,j\colon p \leq i < q \wedge p \leq j < q\colon A(i) - A(j) \leq 1)$$

it was requested to find a statement list $S$ such that

$$
\begin{array}{l}
|[N\colon int\{N \geq 1\} \\
;A(i\colon 0 \leq i < N)\colon \textbf{array of } int \\
;|[c\colon int \\
\quad ;S \\
\quad \{c = (\mathbf{MAX}\ p,q\colon 0 \leq p < q \leq N \wedge SS(p,q)\colon q - p)\} \\
\quad ]| \\
]|
\end{array}
$$

Following the same line of reasoning we employed when solving Exercise 0, we adopt as the invariant

$P0:\quad c = (\mathbf{MAX}\ p,q\colon 0 \leq p < q \leq n \wedge SS(p,q)\colon q - p) \wedge 1 \leq n \leq N$

and we consider $P0^n_{n+1}$.

$P0^n_{n+1}:\quad c = (\mathbf{MAX}\ p,q\colon 0 \leq p < q \leq n+1 \wedge SS(p,q)\colon q - p) \wedge 1 \leq n+1 \leq N.$

The first conjunct may by written as

$$c = (\mathbf{MAX}\ p,q\colon 0 \leqslant p < q \leqslant n \wedge SS(p,q)\colon q-p)$$
$$\mathbf{max}(\mathbf{MAX}\ p\colon 0 \leqslant p < n+1 \wedge SS(p,n+1)\colon n+1-p)$$

or

$$c = (\mathbf{MAX}\ p,q\colon 0 \leqslant p < q \leqslant n \wedge SS(p,q)\colon q-p)$$
$$\mathbf{max}(n+1-(\mathbf{MIN}\ p\colon 0 \leqslant p < n+1 \wedge SS(p,n+1)\colon p)).$$

This suggests extending the invariant with

$P1$: $\quad d = (\mathbf{MIN}\ p\colon 0 \leqslant p < n \wedge SS(p,n)\colon p).$

Thus far our solution is identical to that of Exercise 0. Now we take the definition of smoothness into account. Smooth segments come in two kinds. Define

$$SS0(p,n) \equiv (\mathbf{A}\ i\colon p \leqslant i < n\colon -1 \leqslant A(i) - A(n-1) \leqslant 0),$$
$$SS1(p,n) \equiv (\mathbf{A}\ i\colon p \leqslant i < n\colon 0 \leqslant A(i) - A(n-1) \leqslant 1).$$

Then

$$SS(p,n) \equiv (SS0(p,n) \vee SS1(p,n)).$$

We 'split' $P1$ into $Q0$ and $Q1$:

$Q0$: $\quad d0 = (\mathbf{MIN}\ p\colon 0 \leqslant p < n \wedge SS0(p,n)\colon p),$

$Q1$: $\quad d1 = (\mathbf{MIN}\ p\colon 0 \leqslant p < n \wedge SS1(p,n)\colon p).$

Consider, for $n \geqslant 1$, $Q0^n_{n+1}$.

$Q0^n_{n+1}$: $\quad d0 = (\mathbf{MIN}\ p\colon 0 \leqslant p < n+1 \wedge SS0(p,n+1)\colon p).$

Since $SS0(n,n+1)$ holds, the above may be written as

$$d0 = (\mathbf{MIN}\ p\colon 0 \leqslant p < n \wedge SS0(p,n+1)\colon p)\ \mathbf{min}\ n.$$

Furthermore,

$$SS0(p,n+1) \equiv (\mathbf{A}\ i\colon p \leqslant i < n+1\colon -1 \leqslant A(i) - A(n) \leqslant 0)$$
$$\equiv (\mathbf{A}\ i\colon p \leqslant i < n\colon -1 \leqslant A(i) - A(n) \leqslant 0).$$

If $A(n-1) - A(n) = 0$, we have $SS0(p,n+1) \equiv SS0(p,n)$. If $A(n-1) - A(n) = -1$,

$$SS0(p,n+1) \equiv (\mathbf{A}\ i\colon p \leqslant i < n\colon -1 \leqslant A(i) - A(n-1) - 1 \leqslant 0)$$
$$\equiv SS1(p,n).$$

If $A(n-1) - A(n) < -1 \vee A(n-1) - A(n) > 0$, we have $\neg SS0(n-1,n+1)$ and, hence, $Q0^n_{n+1} \equiv (d0 = n)$.

Thus, for $n \geq 1$, $Q0^n_{n+1}$ is equivalent to

$$d0 = \begin{cases} (\mathbf{MIN}\ p\colon 0 \leq p < n \wedge SS0(p,n)\colon p) & \text{if } A(n-1)-A(n)=0, \\ (\mathbf{MIN}\ p\colon 0 \leq p < n \wedge SS1(p,n)\colon p) & \text{if } A(n-1)-A(n)=-1, \\ n & \text{otherwise.} \end{cases}$$

Likewise, for $n \geq 1$, $Q1^n_{n+1}$ is equivalent to

$$d1 = \begin{cases} (\mathbf{MIN}\ p\colon 0 \leq p < n \wedge SS1(p,n)\colon p) & \text{if } A(n-1)-A(n)=0, \\ (\mathbf{MIN}\ p\colon 0 \leq p < n \wedge SS0(p,n)\colon p) & \text{if } A(n-1)-A(n)=1, \\ n & \text{otherwise.} \end{cases}$$

If $Q0^n_{n+1} \wedge Q1^n_{n+1}$ holds $P0^n_{n+1}$ may be written as

$$c = (\mathbf{MAX}\ p,q\colon 0 \leq p < q \leq n \wedge SS(p,q)\colon q-p)$$
$$\mathbf{max}(n+1-d0)\ \mathbf{max}(n+1-d1).$$

We thus obtain as our solution

$S$: $\;\|[n,d0,d1\colon int$

$\quad ; n,c,d0,d1 := 1,1,0,0\ \{P0 \wedge Q0 \wedge Q1\}$

$\quad ;$ **do** $n \neq N$

$\qquad \rightarrow$ **if** $A(n-1)-A(n) < -1 \rightarrow d0,d1 := n,n$

$\qquad\quad \square\ A(n-1)-A(n) = -1 \rightarrow d0,d1 := d1,n$

$\qquad\quad \square\ A(n-1)-A(n) = 0 \rightarrow$ **skip**

$\qquad\quad \square\ A(n-1)-A(n) = 1 \rightarrow d0,d1 := n,d0$

$\qquad\quad \square\ A(n-1)-A(n) > 1 \rightarrow d0,d1 := n,n$

$\qquad\quad$ **fi**$\{P0 \wedge Q0^n_{n+1} \wedge Q1^n_{n+1}\}$

$\qquad ;\ c := c\ \mathbf{max}(n+1-d0)\ \mathbf{max}\ (n+1-d1)\ \{P0^n_{n+1} \wedge Q0^n_{n+1} \wedge Q1^n_{n+1}\}$

$\qquad ;\ n := n+1\ \{P0 \wedge Q0 \wedge Q1\}$

$\quad$ **od**

$\quad ]\|$

I owe this solution to W.H.J. Feijen.

There is a different solution whose computation time is also linear in $N$. One may arrive at that solution by sticking to P1 and extending the invariant $P0 \wedge P1$ with $P2$.

$P2$: $\quad mim = (\mathbf{MIN}\ p\colon d \leq p < n\colon A(p))$

$\qquad \wedge\ mam = (\mathbf{MAX}\ p\colon d \leq p < n\colon A(p)).$

We then obtain the following solution.

$S'$: $\quad$ |[$n,d,mim,mam$: *int*

; $n,c,d,mim,mam := 1,1,0,A(0),A(0)$ $\{P0 \wedge P1 \wedge P2\}$

; **do** $n \neq N$

$\rightarrow$ **if** $A(n) \leqslant mim+1 \wedge A(n) \geqslant mam-1$

$\rightarrow mim, mam := mim$ **min** $A(n)$, $mam$ **max** $A(n)$

$\square\, A(n) > mim+1 \vee A(n) < mam-1$

$\rightarrow d, mim, mam := n, A(n), A(n)$

; **do** $d \neq 0$ **cand** $(A(d-1) \leqslant mim+1 \wedge A(d-1) \geqslant mam-1)$

$\rightarrow d, mim, mam := d-1, mim$ **min** $A(d-1)$, $mam$ **max** $A(d-1)$

**od**

**fi** $\{P0 \wedge P1^{n}_{n+1} \wedge P2^{n}_{n+1}\}$

; $c := c$ **max**$(n+1-d)$ $\{P0^{n}_{n+1} \wedge P1^{n}_{n+1} \wedge P2^{n}_{n+1}\}$

; $n := n+1$ $\{P0 \wedge P1 \wedge P2\}$

**od**

]|

The Boolean connectives **cand** and **cor**, the former of which occurs in $S'$, are the conditional $\wedge$ and $\vee$. These are asymmetric operators satisfying

($true$ **cand** $B$) $\equiv$ $B$

($false$ **cand** $B$) $\equiv$ $false$ (even if $B$ is undefined)

($B0$ **cor** $B1$) $\equiv$ $\neg(\neg B0$ **cand** $\neg B1)$

One may think that the inner repetition destroys the linearity of the solution. This, however, is not the case, as I will show for a generalization of the above algorithm. Let $k$ be a natural number, $k \geqslant 0$. We call a segment $A(i\colon p \leqslant i < q)$ a $k$-smooth segment if

$$(\mathbf{A}\ i,j\colon p \leqslant i < q \wedge p \leqslant j < q\colon A(i) - A(j) \leqslant k).$$

By replacing each of the (six) occurrences of 1 in the guards of $S'$ by $k$ we obtain a program for the computation of the maximal length of any $k$-smooth segment. The program thus obtained is due to Henk Middendorp. There is no such simple generalization for our former solution.

We call a $k$-smooth segment that is not contained in a longer such segment maximal. Since the second guarded command of the alternative construct is selected at most once for every maximal segment, the total number of steps of the inner repetition is bounded by the sum of the lengths of the maximal segments. Since two non-disjoint maximal segments have distinct minimal values, any $A(i)$ belongs to at most $k+1$ maximal segments. As a consequence, the sum of the lengths of the maximal segments is at most $(k+1)\cdot N$, which shows the worst-case complexity of the program to be $O(k\cdot N)$ and that of $S'$, consequently, $O(N)$. The above is an outline of a proof given by Jan Tijmen Udding.

There is another remark that applies to $S'$. The number of steps of the outer repetition may be decreased by taking into account whether the segment $A(i: d\leqslant i<N)$ is of sufficient length to accommodate a smooth segment of a length exceeding $c$. Thus, its guard may be strengthened to $N-d>c$. The repetition has $c\geqslant n-d$ as an invariant. Since

$$N-d>c \wedge c\geqslant n-d \Rightarrow n<N$$

the replacement of $n\neq N$ by $N-d>c$ is indeed a strengthening.

*Solution of Exercise 2 (Fibolucci)*

With the function $f$ defined by

$$f(0)=1,$$

$$f(1)=1,$$

$$f(i+2)=X\cdot f(i)+Y\cdot f(i+1) \quad \text{for } i\geqslant 0$$

the exercise was to find a statement list $S$ such that

$|[N,X,Y\colon int\{N\geqslant 1\}$

$;|[a\colon int$

$\quad ;S$

$\quad \{a=(\mathbf{S}\, i\colon 0\leqslant i\leqslant N\colon f(i)\cdot f(N-i))\}$

$\quad ]|$

$]|$

In the standard fashion we derive an invariant $P0$ by replacing in the postcondition the constant $N$ by a suitably bounded variable $n$.

$P0$: $\quad a=(\mathbf{S}\, i\colon 0\leqslant i\leqslant n\colon f(i)\cdot f(n-i)) \wedge 1\leqslant n\leqslant N.$

Again we focus our attention on the first conjunct of $P0^n_{n+1}$, i.e. on

$$a=(\mathbf{S}\, i\colon 0\leqslant i\leqslant n+1\colon f(i)\cdot f(n+1-i))$$

This may be written as

$$a=(\mathbf{S}\, i\colon 0\leqslant i\leqslant n-1\colon f(i)\cdot f(n+1-i))$$
$$+f(n)\cdot f(1)+f(n+1)\cdot f(0)$$

or

$$a=(\mathbf{S}\ i: 0 \leqslant i \leqslant n-1: f(i)\cdot(X\cdot f(n-1-i)+Y\cdot f(n-i)))$$
$$+f(n)+f(n+1)$$

or

$$a=X\cdot(\mathbf{S}\ i: 0 \leqslant i \leqslant n-1: f(i)\cdot f(n-1-i))$$
$$+\ Y\cdot(\mathbf{S}\ i: 0 \leqslant i \leqslant n: f(i)\cdot f(n-i))-f(n))$$
$$+f(n)+f(n+1)$$

The formula above suggests an extension of the invariant with

$P1$: $\quad b=(\mathbf{S}\ i: 0 \leqslant i \leqslant n-1: f(i)\cdot f(n-1-i))$.

Observe that

$$P1^{n}_{n+1} \equiv (b=(\mathbf{S}\ i: 0 \leqslant i \leqslant n: f(i)\cdot f(n-i))).$$

We, furthermore, add to the invariant

$P2$: $\quad g=f(n-1) \wedge h=f(n)$.

For $P2^{n}_{n+1}$ we find

$$g=f(n) \wedge h=f(n+1)=X\cdot f(n-1)+Y\cdot f(n).$$

Thus, our solution becomes

$S$: $\quad$ |[ $n,b,g,h$: *int*

; $n,a,b,g,h := 1,2,1,1,1 \quad \{P0 \wedge P1 \wedge P2\}$

; **do** $n \neq N$

$\rightarrow g,h := h, X*g+Y*h \quad \{P0 \wedge P1 \wedge P2^{n}_{n+1}\}$

; $a,b := X*b+Y*(a-g)+g+h, a$

$\{P0^{n}_{n+1} \wedge P1^{n}_{n+1} \wedge P2^{n}_{n+1}\}$

; $n := n+1 \quad \{P0 \wedge P1 \wedge P2\}$

**od**

]|